On selfinjective algebras of stable dimension zero

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For more details, please refer to the preprint arXiv:1004.1723v1.

Preliminaries

Throughout this talk,

k : an algebraically closed field.

A : a non-semisimple connected *selfinjective* finite-dimensional k-algebra,

$(:\Longleftrightarrow {}_A A \cong {}_A D(A) \Leftrightarrow A_A \cong D(A)_A$, where $D = \mathrm{Hom}_k(\text{-}, k))$.

$\mathrm{mod}\, A$: the abelian category of finitely generated right A-modules.

$\mathrm{ind}\, A$: the full subcategory of $\mathrm{mod}\, A$ consisting of indecomposable modules.

$\underline{\mathrm{mod}}\, A$: the stable module category of A,

$\mathrm{obj}(\underline{\mathrm{mod}}\, A) := \mathrm{obj}(\mathrm{mod}\, A)$ and $\underline{\mathrm{mod}}\, A(X, Y) := \mathrm{mod}\, A(X, Y)/\, \mathrm{P}(X, Y)$

where $\mathrm{P}(X, Y) := \{ f \mid$
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & \exists P \arrow[ur] & \end{tikzcd}$$
with P projective$\}$.

$$\Omega_A : \underline{\mathrm{mod}}\, A \xrightarrow{\sim} \underline{\mathrm{mod}}\, A : \text{ a syzygy functor,}$$

$$i.e., \ \Omega_A(X) := \text{the kernel of a projective cover } P_X \to X.$$

Note that $\underline{\mathrm{mod}}\, A$ is a triangulated category with a shift functor Ω_A^{-1}.

Definition. The *stable dimension* of A is defined as follows:

$$\begin{aligned} \mathrm{stab}\,.\,\dim A &:= \dim(\underline{\mathrm{mod}}\, A) \qquad\qquad \text{(see Rouquier[5])} \\ &:= \min\{n > 0 | \langle M \rangle_{n+1} = \underline{\mathrm{mod}}\, A \ \text{ for } \ \exists M \in \underline{\mathrm{mod}}\, A\} \end{aligned}$$

where $\langle M \rangle_{n+1}$ is defined inductively:

$$\begin{aligned} &\text{for } \ n = 0, \qquad \langle M \rangle_1 := \mathrm{add}\{\Omega_A^i M | i \in \mathbb{Z}\}, \ \text{ and} \\ &\text{if } \ n > 0, \qquad \langle M \rangle_{n+1} := \mathrm{add}\{M_{n+1} | \exists \triangle : M_n \to M_{n+1} \to M_1 \to \ , \\ &\qquad\qquad\qquad \text{where } \ M_n \in \langle M \rangle_n \text{ and } M_1 \in \langle M \rangle_1\}. \end{aligned}$$

Remark. $\underline{\mathrm{mod}}\, A \overset{\triangle}{\cong} \underline{\mathrm{mod}}\, A' \Rightarrow \mathrm{stab}\,.\,\dim A = \mathrm{stab}\,.\,\dim A'$.

Motivation

Proposition 1. ($Rouquier$[4], 2006 **cf**.$Auslander$[1]) *Let B be a non-semisimple selfinjective algebra over a field. Then*

$$\mathrm{LL}(B) \geq \operatorname{rep.dim} B \geq \operatorname{stab.dim} B + 2$$

where $\operatorname{rep.dim} B := \min\{\operatorname{gl.dim} \operatorname{End}_B(M) | M \in \operatorname{mod} B \text{ is a generator and cogenerator}\}$,

$\mathrm{LL}(B) := \min\{r | \operatorname{rad}(B)^r = 0\}$.

Remark. $\operatorname{stab.dim} B$ is always finite.

Theorem 2. ($Auslander$[1], 1971) *For any artin algebra Λ,*

$$\Lambda : \textit{representation-finite} \Longleftrightarrow \operatorname{rep.dim} \Lambda \leq 2.$$

Observation. B : representation-finite $\Longrightarrow \operatorname{stab.dim} B = 0$.

Our result

The converse is also true

for selfinjective algebras over an algebraically closed field.

Notation

Γ_A : the Auslander-Reiten quiver of A,

the vertices are the isoclasses $[X]$ of $X \in \operatorname{ind} A$, the arrows are irreducible maps.

$\tau_A = D\,\mathrm{Tr}$: the Auslander-Reiten translation.

$\Rightarrow \tau_A \cong \Omega_A^2 \circ \nu_A \cong \nu_A \circ \Omega_A^2$

where $\nu_A := D\,\mathrm{Hom}_A(\text{-}, A)$ is a Nakayama functor.

$M \in \Gamma_A$ is τ_A-periodic $:\Longleftrightarrow \tau_A^n M = M$ for $\exists n > 0$

Proposition 3. (Y)

$$\sharp\{\Omega_A\text{-}orbits\ in\ \Gamma_A\} < \infty \Longrightarrow \begin{cases} (1)\ \sharp\{\tau_A\text{-}orbits\ in\ \Gamma_A\} < \infty,\ or \\ (2)\ \Gamma_A\ \ is\ finite. \end{cases}$$

Proposition 4. (*Liu*[*3*], 1992) *For a finite-dimensional algebra* Λ *over an algebraically closed field,*

$$\Lambda\ \ is\ representation\text{-}finite \Longleftrightarrow\ \sharp\{\tau_\Lambda\text{-}orbits\ in\ \Gamma_\Lambda\} < \infty$$

.

Theorem 5. (Y)

$$A \textit{ is representation-finite} \iff \operatorname{stab.dim} A = 0.$$

Proof. It is sufficiently to show the if part.

$$\begin{array}{rcl}
\operatorname{stab.dim} A = 0 & \Leftrightarrow & \underline{\mathrm{mod}}\, A = \mathrm{add}\{\Omega_A^i M | i \in \mathbb{Z}\} \text{ for } \exists M \in \underline{\mathrm{mod}}\, A \\
 & \Leftrightarrow & \sharp\{\Omega_A\text{-orbits in } \Gamma_A\} < \infty \\
\text{By Proposition3} & \Rightarrow & \sharp\{\tau_A\text{-orbits in } \Gamma_A\} < \infty, \text{ or} \\
 & & \Gamma_A \text{ is finite.}
\end{array}$$

Even if $\sharp\{\tau_A\text{-orbits in } \Gamma_A\} < \infty$, then A is representation-finite by Proposition 4. $\square$

Corollary 6. $\operatorname{rep.dim} A = 3 \Rightarrow \operatorname{stab.dim} A = 1$.

Conjecture. Λ *tame* (*not necessarily selfinjective*) $\Longrightarrow \operatorname{rep.dim} \Lambda \leq 3$.

$$\not\Longleftarrow$$

Any wild hereditary algebra is a counter example.

Hope. A *tame selfinjective* $\Longrightarrow \operatorname{stab.dim} A \leq 1$.

$$\not\Longleftarrow$$

Any wild selfinjective algebra with radical cube zero is a counter example.

Example

Let $A = kQ/I$, where Q is the quiver

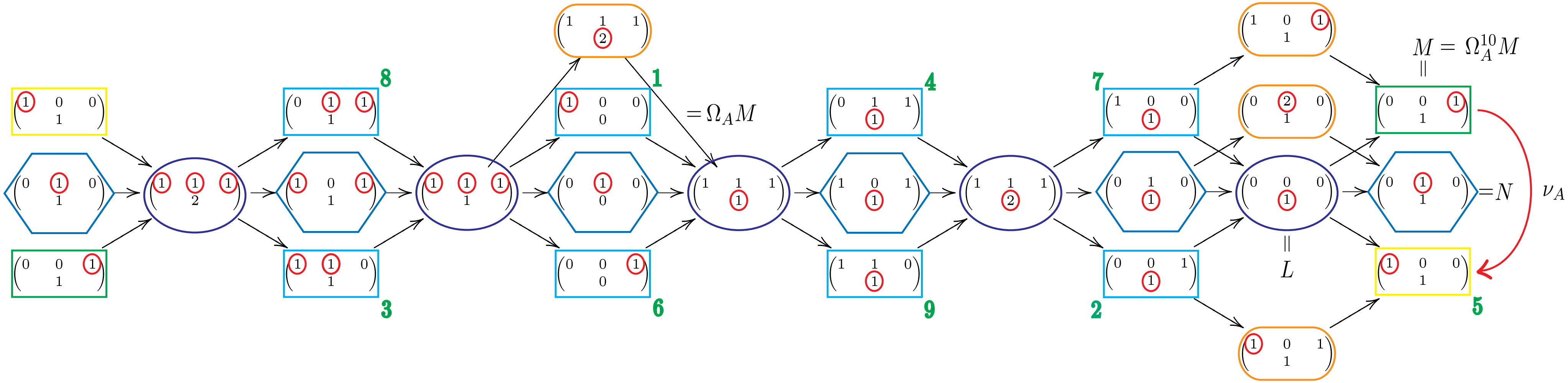
$$Q: \quad \begin{tikzcd} & 2 \arrow[d, shift right, "\alpha_2"'] & \\ & 0 \arrow[u, shift right, "\beta_2"'] \arrow[dl, shift right, "\beta_1"'] \arrow[dr, shift right, "\beta_3"'] & \\ 1 \arrow[ur, shift right, "\alpha_1"'] & & 3 \arrow[ul, shift right, "\alpha_3"'] \end{tikzcd}$$

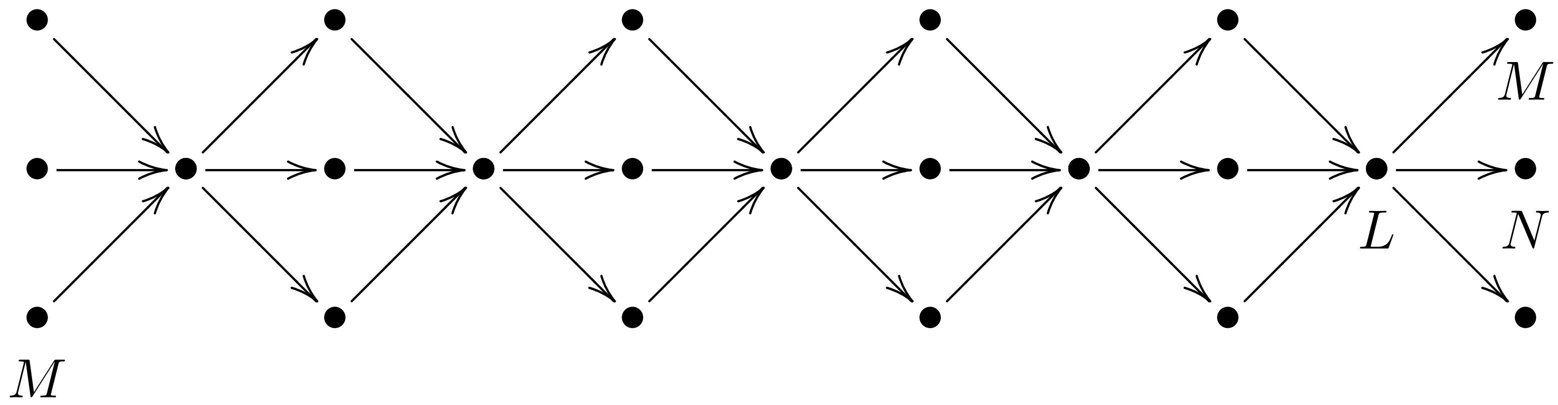
and the ideal I is generated by

$$\alpha_1\beta_1 - \alpha_2\beta_2,\ \alpha_2\beta_2 - \alpha_3\beta_3,\ \beta_1\alpha_1,\ \beta_2\alpha_1,\ \beta_1\alpha_2,\ \beta_3\alpha_2,\ \beta_2\alpha_3,\ \beta_3\alpha_3.$$

Then A is a selfinjective algebra of type D_4.

Thus, the Auslander-Reiten quiver Γ_A of A is of the form:





We set $X = M \oplus N \oplus L$. Then $\underline{\mathrm{mod}}\, A = \mathrm{add}\{\Omega_A^i X | i \in \mathbb{Z}\}$ and hence $\mathrm{stab\,.\,dim}\, A = 0$.

References

[1] M. Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes, London, 1971.

[2] M. Auslander, I. Reiten and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics **36**, Cambridge, 1995.

[3] S. Liu, *Degrees of irreducible maps and the shapes of Auslander-Reiten quivers*, J. London Math. Soc. (2) **45** (1992), no. 1, 32-54.

[4] R. Rouquier, *Representation dimension of exterior algebras*, Invent. Math. **165** (2006), no. 2, 357-367.

[5] R. Rouquier, *Dimensions of triangulated categories*, J. of K-theory **1** (2008), no. 2, 193-256 and errata, 257-258.