On the Glauberman-Watanabe corresponding blocks as bimodules

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1. Let p be a prime. Let $\mathcal{O}$ be a complete discrete valuation ring having an algebraically closed residue field of characteristic p and having a quotient field $\mathcal{K}$ of characteristic zero which will be assumed to be "large enough". Below, modules are finitely generated $\mathcal{O}$-free modules.

2. Let b be a (p-)*block* (*idempotent*) of a finite group G (that is, a primitive idempotent of the center $Z(\mathcal{O}G)$ of the group algebra $\mathcal{O}G$ of G over $\mathcal{O}$) with a defect group D. (Here, a *defect group* of b is a minimal subgroup D of G such that any $\mathcal{O}Gb$-module M is isomorphic to a direct summand of $T{\uparrow}_D^G$ for some $\mathcal{O}D$-module T.) By the action $g_1 \cdot x \cdot g_2 = g_1 x g_2$ where $g_1, g_2 \in G$ and $x \in \mathcal{O}G$, *block* (*algebra*) $\mathcal{O}Gb$ is an indecomposable $(\mathcal{O}G, \mathcal{O}G)$-bimodule.

3. As is usual way, we do not distinguish an $(\mathcal{O}G_1, \mathcal{O}G_2)$-bimodule X and an $\mathcal{O}[G_1 \times G_2]$-module X for two groups G_1 and G_2 by $g_1 \cdot m \cdot g_2 = (g_1, g_2^{-1}) \cdot x$ where $g_1 \in G_1$, $g_2 \in G_2$ and $x \in X$. Then $\mathcal{O}Gb$ is an indecomposable $\mathcal{O}[G \times G]$-module with a vertex $\Delta D = \{(d,d) \,|\, d \in D\}$, see [NT]. (Here, a *vertex* of an indecomposable $\mathcal{O}L$-module N for a group L is a minimal subgroup P of L such that N is isomorphic to a direct summand of $U{\uparrow}_P^L$ for some $\mathcal{O}P$-module U).

4. For a subgroup H of G containing $N_G(D)$, *Brauer corresponding* block (see [NT]) $\mathcal{O}Hc$ of H viewed as as a bimodule can be characterized as a unique direct summand of $\mathcal{O}Gb{\downarrow}_{H \times H}^{G \times G}$ with a vertex ΔD, and $\mathcal{O}Gb$ viewed as a bimodule can be characterized as a unique direct summand of $\mathcal{O}Hc{\uparrow}_{H \times H}^{G \times G}$ with a vertex ΔD, see [NT]. That is, Brauer corresponding blocks viewed as bimodules are the *Green corresponding* (see [NT]) modules.

5. Let q be a prime such that $q \not| |G|$. Let S be a cyclic group of order q acting on G. Then with this action, we can consider the semi-direct product of G and S, denoted by GS. Let $\ddot{S}$ be a subgroup of $S \times S (\subset GS \times GS)$ such that the canonical projections $S \times S \to S \times 1$ and $S \times S \to 1 \times S$ are isomorphisms. Denote by G^S the centralizer of S in G.

6. Glauberman showed that for an S-invariant irreducible $\mathcal{K}$-character χ, there is a unique irreducible $\mathcal{K}$-character $\pi(G,S)(\chi)$ of G^S such that $\pi(G,S)(\chi)$ is a constituent of $\chi{\downarrow}_{G^S}^G$ with a multiplicity not divisible by q (in fact, its multiplicity is ± 1 modulo q), and $\pi(G,S)$ gives a one-to-one correspondence, called the *Glauberman correspondence*, between the set $\mathrm{Irr}(G)^S$ of S-invariant irreducible $\mathcal{K}$-characters of G and the set $\mathrm{Irr}(G^S)$ of irreducible $\mathcal{K}$-characters of G^S. Note that χ is a unique S-invariant constituent of $\pi(G,S)(\chi){\uparrow}_{G^S}^G$ with a multiplicity not divisible by q. For a precise statement, see [G].

7. Assume that b is S-invariant and a defect group D of b is centralized by S. Then Watanabe showed that all irreducuble $\mathcal{K}$-characters in b ($\chi \in \mathrm{Irr}(G)$ is *in* b if a $\mathcal{K}G$-module affording χ is not annihilated by b) are S-invariant, that is, $\mathrm{Irr}(b) = \mathrm{Irr}(b)^S$, and there is a block $w(b)$ of G^S, called the *Glauberman-Watanabe corresponding* block of b, with a defect group D such that $\mathrm{Irr}\big(w(b)\big) = \{\pi(G,S)(\chi) \,|\, \chi \in \mathrm{Irr}(b)\}$. For a precise statement, see [W].

8. As in **4** for the Brauer corresponding blocks, we can characterize the Glauberman-Watanabe corresponding blocks viewed as bimodules in terms of a vertex and a multiplicity as a direct summand of a restricted or an induced module from the block. Note that when G^S contains $N_G(D)$, the Glauberman-Watanabe corresponding block coincids with the Brauer corresponding block and the characterization in Theorem below is compatible with the one for the Brauer corresponding blocks in **4**.

Theorem

(1) $\mathcal{O}G^S w(b)$ *is a unique indecomposable direct summand of* $\mathcal{O}Gb{\downarrow}^{G\times G}_{G^S\times G^S}$ *with a vertex* ΔD *and with a multiplicity not divisible by* q*. In fact, its multiplicity is 1 modulo* q*.*

(2) $\mathcal{O}Gb$ *is a unique* $\ddot{S}$*-invariant indecomposable direct summand of* $\mathcal{O}G^S w(b){\uparrow}^{G\times G}_{G^S\times G^S}$ *with a vertex* ΔD *and with a multiplicity not divisible by* q*. In fact, its multiplicity is 1 modulo* q*.*

References

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