FROBENIUS CONDITION ON A PRETRIANGULATED CATEGORY

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Abstract. As shown by Happel, from any Frobenius exact category, we can construct a triangulated category as a stable category. On the other hand, it was shown by Iyama and Yoshino that if a pair of subcategories $\mathcal{D} \subseteq \mathcal{Z}$ in a triangulated category satisfies certain conditions (i.e., $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair), then $\mathcal{Z}/\mathcal{D}$ becomes a triangulated category. In this talk, we introduce a simultaneous generalization of these two constructions.

1. Introduction and Preliminaries

Throughout this talk, we fix an additive category $\mathcal{C}$. Any subcategory of $\mathcal{C}$ will be assumed to be full, additive and replete. A subcategory is called *replete* if it is closed under isomorphisms.

When we say $\mathcal{Z}$ is an exact category, we only consider an extension-closed subcategory of an abelian category.

For any category $\mathcal{K}$, we write abbreviately $K \in \mathcal{K}$, to indicate that K is an object of $\mathcal{K}$. For any $K, L \in \mathcal{K}$, let $\mathcal{K}(K, L)$ denote the set of morphisms from K to L. If $\mathcal{M}, \mathcal{N}$ are full subcategories of $\mathcal{K}$, then $\mathcal{K}(\mathcal{M}, \mathcal{N}) = 0$ means that $\mathcal{K}(M, N) = 0$ for any $M \in \mathcal{M}$ and $N \in \mathcal{N}$. Similarly, $\mathcal{K}(K, \mathcal{N}) = 0$ means $\mathcal{K}(K, N) = 0$ for any $N \in \mathcal{N}$.

If $\mathcal{K}$ is an additive category and $\mathcal{L}$ is a full additive replete subcategory which is closed under finite direct summands, then $\mathcal{K}/\mathcal{L}$ denotes the quotient category of $\mathcal{K}$ by the ideal generated by $\mathcal{L}$. The image of $f \in \mathcal{K}(X, Y)$ will be denoted by $\underline{f} \in \mathcal{K}/\mathcal{L}(X, Y)$.

As shown by Happel [H], If we are given a Frobenius exact category $\mathcal{E}$, then the stable category $\mathcal{E}/\mathcal{I}$, where $\mathcal{I}$ is the full subcategory of injectives, carries a structure of a triangulated category.

On the other hand, it was shown by Iyama and Yoshino that if $\mathcal{D} \subseteq \mathcal{Z}$ is a pair of subcategories in a triangulated category $\mathcal{C}$ such that $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair, then the quotient category $\mathcal{Z}/\mathcal{D}$ becomes a triangulated category. In this talk, we make a simultaneous generalization of these two constructions, by using a slight modification of a *pretriangulated category* in [BR].

2. Generalization of the Frobenius condition

Definition 2.1. Let $\Sigma\colon \mathcal{C} \to \mathcal{C}$ be an additive endofunctor, and let $\mathcal{RT}(\mathcal{C}, \Sigma)$ be the category of diagrams of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A.$$

A pair $(\Sigma, \rhd)$ of Σ and a full replete subcategory $\rhd \subseteq \mathcal{RT}(\mathcal{C}, \Sigma)$ is called a *right triangulation* on $\mathcal{C}$ if it satisfies similar conditions to those for a triangulated category, except that Σ is not necessarily an equivalence. A left triangulation $(\Omega, \lhd)$ is defined dually.

If we are given an adjoint isomorphism $\psi\colon \mathcal{C}(\Omega(-), -) \xrightarrow{\cong} \mathcal{C}(-, \Sigma(-))$ satisfying some gluing conditions, we call a 5-tuple $(\Sigma, \Omega, \rhd, \lhd, \psi)$ a *pseudo-triangulation* on $\mathcal{C}$. We call the 6-tuple $(\mathcal{C}, \Sigma, \Omega, \rhd, \lhd, \psi)$ a *pseudo-triangulated category*, and often represent it simply by $\mathcal{C}$.

Example 2.2. Let $(\mathcal{C}, \Sigma, \Omega, \rhd, \lhd, \psi)$ be a pseudo-triangulated category.

(1) $\mathcal{C}$ is an abelian category if and only if $\Sigma = \Omega = 0$.
(2) $\mathcal{C}$ is a triangulated category if and only if Σ is the quasi-inverse of Ω and ψ is the one induced from the isomorphism $\Sigma \circ \Omega \cong \mathrm{Id}_{\mathcal{C}}$.

Definition 2.3. A sequence in $\mathcal{C}$

$$\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

is called an *extension* if it satisfies

$$(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A) \in \rhd,$$
$$(\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C) \in \lhd,$$
$$h = -\psi_{C,A}(e).$$

Example 2.4. The notion of an extension becomes as follows in the two cases of Example 2.2.

(1) If $\Sigma = \Omega = 0$ and $\mathcal{C}$ is abelian, then an extension is nothing other than a short exact sequence.
(2) If $\mathcal{C}$ is a triangulated category as in Example 2.2, then an extension is nothing other than a distinguished triangle.

Definition 2.5. A subcategory $\mathcal{Z} \subseteq \mathcal{C}$ is said to be *extension-closed* if for any extension in $\mathcal{C}$

$$\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

$X, Z \in \mathcal{Z}$ implies $Y \in \mathcal{Z}$.

Injective objects and projective objects are defined using extensions, similarly to those in an exact category. We denote the full subcategory of injectives (resp. projectives) by $\mathcal{I}$ (resp. $\mathcal{P}$).

If $\mathcal{Z} \subseteq \mathcal{C}$ is an exact category with $\mathcal{C}$ abelian as in Example 2.4, then the definitions of injectives and projectives agree with those in an exact category.

Definition 2.6. $\mathcal{Z}$ is *Frobenius* if it has enough injectives and projectives, and moreover $\mathcal{I} = \mathcal{P}$.

Example 2.7.

(1) If $\mathcal{Z} \subseteq \mathcal{C}$ is an exact category, then $\mathcal{Z}$ is Frobenius if and only if $\mathcal{Z}$ is Frobenius as an exact category.
(2) If $\mathcal{C}$ is a triangulated category and if $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair, then $\mathcal{Z}$ is Frobenius.

More precisely, we have the following. Remark that those $\mathcal{D}$ such that $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair, is unique if it exists.

Corollary 2.8. *For any $\mathcal{Z}$, the following are equivalent.*

(1) *$\mathcal{Z}$ is Frobenius, and $\mathcal{C}(\Omega\mathcal{Z},\mathcal{I}) = \mathcal{C}(\mathcal{I},\Sigma\mathcal{Z}) = 0$.*
(2) *$(\mathcal{Z},\mathcal{Z})$ is an $\mathcal{I}$-mutation pair.*

In the rest, $\mathcal{C}$ is assumed to satisfy the following additional condition. Remark that this condition is trivially satisfied in the two cases in Example 2.4.

Condition 2.9. Let

$$\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$
$$\Omega C' \xrightarrow{e'} A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$$

be extensions.

(AC1) If $c \in \mathcal{C}(C, C')$ satisfies $h' \circ c = 0$ and $c \circ g = 0$, then there exists $c' \in \mathcal{C}(C, B')$ such that $g' \circ c' = c$.
(AC2) If $a \in \mathcal{C}(A, A')$ satisfies $f' \circ a = 0$ and $a \circ e = 0$, then there exists $a' \in \mathcal{C}(B, A')$ such that $a' \circ f = a$.

As a main theorem, we give a triangulation on $\mathcal{Z}/\mathcal{I}$. First, we construct the shift functor.

Definition 2.10. Assume $\mathcal{Z}$ is Frobenius. For any $X \in \mathcal{Z}$, take an extension

$$\Omega S_X \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X$$

in $\mathcal{Z}$, with $I_X \in \mathcal{I}$. Define $S(X) = SX$ to be the image of S_X in $\mathcal{Z}/\mathcal{I}$. We can show $S\colon \mathcal{Z}/\mathcal{I} \to \mathcal{Z}/\mathcal{I}$ gives an additive endoequivalence.

Secondly, the class of distinguished triangles are given as follows.

Definition 2.11. Let $\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be any extension in $\mathcal{Z}$, and take an extension $\Omega S_X \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} F_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X$ in $\mathcal{Z}$ where $I_X \in \mathcal{I}$.

If there exist $p \in \mathcal{Z}(Y, I_X)$ and $q \in \mathcal{Z}(Z, S_X)$ satisfying

$$p \circ f = \alpha_X, \quad q \circ g = \beta_X \circ p, \quad \gamma_X \circ q = h,$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "g"] \arrow[d, "p"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, "h"] \arrow[d, "q"] \arrow[dr, phantom, "\circlearrowright"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "\alpha_X"'] & I_X \arrow[r, "\beta_X"'] & S_X \arrow[r, "\gamma_X"'] & \Sigma X \end{tikzcd}$$

then we call the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{q} SX$ a *standard triangle*. We define the class of distinguished triangles $\triangle$ to be the category of triangles $X \to Y \to Z \to SZ$ in $\mathcal{Z}/\mathcal{I}$, which are isomorphic to standard triangles.

Then we have the following.

Theorem 2.12. *Let $\mathcal{C}$ be a pseudo-triangulated category satisfying Condition 2.9, and let $\mathcal{Z} \subseteq \mathcal{C}$ be an extension-closed subcategory. If $\mathcal{Z}$ is Frobenius, $(\mathcal{Z}/\mathcal{I}, S, \triangle)$ becomes a triangulated category.*

This generalizes Happel's and Iyama-Yoshino's constructions simultaneously.

	$\Sigma = \Omega = 0$	$\Sigma \cong \Omega^{-1}$
Pretriangulated	abelian	triangulated
Extension	short exact sequence	distinguished triangle
Frobenius condition	Frobenius condition	Corollary 2.8
Theorem 2.12	Happel's construction	Iyama-Yoshino's construction

3. Possibility of further generalizations

In [B], for any triangulated category $\mathcal{C}$, Beligiannis showed that if we are given a *proper class of triangles* $\mathcal{E}$ on $\mathcal{C}$ satisfying some conditions similar to the Frobenius condition discussed in section 2, then $\mathcal{C}/\mathcal{P}(\mathcal{E})$ becomes triangulated (Theorem 7.2 in [B]). Here, $\mathcal{P}(\mathcal{E})$ is the subcategory of 'projectives', defined in a similar, but different manner (Definition 4.1 in [B]). With that definition, $\mathcal{P}(\mathcal{E})$ becomes closed under Σ, but this conflicts with Iyama-Yoshino's construction, in which the factoring category $\mathcal{D}$ satisfies $\mathcal{C}(\mathcal{D}, \Sigma\mathcal{D}) = 0$. We wonder if there exists a general construction unifying the construction in [B] and that in this talk.

We also remark that there is another very general construction of a triangulated stable category. In [BM], Beligiannis and Marmaridis constructed a left triangulated category (in the sense of [B] or [BM]) from a pair $(\mathcal{C}, \mathcal{X})$ of an additive category $\mathcal{C}$ and a contravariantly finite subcategory $\mathcal{X}$ assuming some existence condition on kernels (Theorem 2.12 in [BM]). Therefore if $\mathcal{X}$ is functorially finite and satisfies some nice properties, it is expected that this resulting category becomes triangulated. In fact, Happel's construction is one of these cases (Remark 2.14 in [BM]). Although this existence condition is not satisfied by a triangulated category $\mathcal{C}$ unless we replace it by some 'pseudo' one, we hope some unifying construction will be possible.

References

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