On the symmetry of selfinjective dimension

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Abstract

Let R be a commutative Noether ring and A a Noether R-algebra. Set $\Omega = \mathrm{Hom}_R(A, R)$. We will show that proj dim ${}_A\Omega \leq 1$ if and only if proj dim $\Omega_A \leq 1$ under some conditions. Also, we will show that inj dim ${}_AA \leq \dim R + 1$ if and only if inj dim $A_A \leq \dim R + 1$ under some conditions. Assume that proj dim $\Omega_A < \infty$. Let $P^\bullet \to \Omega$ be a projective resolution with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$. We will provide a sufficient condition for $P^\bullet$ to be a partial tilting complex.

1 Introduction and notation

1.1 Notation

Let A be a ring. We denote by Mod-A the category of right A-modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation M_A (resp., ${}_AM$) to stress that the module M considered is a right (resp., left) A-module. We denote by $\mathcal{P}_A$ the full subcategory of Mod-A consisting of finitely generated projective modules. For an additive category $\mathcal{A}$ we denote by $\mathcal{K}(\mathcal{A})$ the homotopy category of cochain complexes over $\mathcal{A}$ and by $\mathcal{K}^{\mathrm{b}}(\mathcal{A})$ the full triangulated subcategories of $\mathcal{K}(\mathcal{A})$ consisting of bounded complexes. We refer to [Gr] for the definition of Grothendieck groups and denote by $\mathrm{K}_0(A)$ the Grothendieck group of A. For an object X in an additive category $\mathcal{A}$ we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of copies of X. We denote by $\mathrm{Hom}^\bullet(-,-)$ the associated single complex of the double hom complex. For a cochain complex $X^\bullet$ over an abelian category $\mathcal{A}$ we denote $\mathrm{H}^i(X^\bullet)$ the ith cohomology of $X^\bullet$. We consider modules as complexes concentrated in degree zero.

Let R be a commutative ring. We denote by $\dim R$ the Krull dimension of R. We denote by $\mathrm{Spec}(R)$ the set of prime ideals of R. For any $\mathfrak{p} \in \mathrm{Spec}(R)$ we denote by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p}$. For any $M \in \text{Mod-}R$ we set $\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$.

1.2 Introduction

Let A be a Noether ring, i.e., A is a left and right Noether ring. Does it hold true that inj dim $_AA < \infty$ implies inj dim $A_A < \infty$? In [Za, Lemma A] Zaks showed that if inj dim $_AA < \infty$ and inj dim $A_A < \infty$ then inj dim $_AA =$ inj dim A_A. This problem is still open.

We will consider the case where R is a commutative Noether ring and A is a Noether R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. Note that A is left and right noetherian. Assume that R and A satisfy the following conditions: (1) $R_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$; (2) $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A, R)$. Then we will show that proj dim $_A\Omega \leq 1$ if and only if proj dim $\Omega_A \leq 1$ (Theorem 2.2). Assume further that R is a Gorenstein local ring. Then we will provide a formula of selfinjective dimension (Proposition 2.3) and show the symmetry of selfinjective dimension (Cororally 2.4). Assume further that $\sup \{\mathrm{ht}\, \mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$. We know from [Ab, Theorem 3.9] that the following are equivalent: (1) inj dim $_AA =$ inj dim $A_A < \infty$; (2) proj dim $\Omega_A =$ proj dim $_A\Omega < \infty$; (3) Ω is a tilting module. Take a projective resolution $P^\bullet \to \Omega$ in mod-A. If $P^\bullet$ is a partial tilting complex, i.e., $P^\bullet$ is a direct summand of a tilting complex, then we can conlude that inj dim $_AA < \infty$ if and only if inj dim $A_A < \infty$ (cf. Proposition 3.11). So we ask when $P^\bullet$ is a partial tilting complex.

Let $P^\bullet \in \mathcal{K}^\mathrm{b}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$. We will provide a sufficient condition for $P^\bullet$ to be a direct summand of a silting complex (Theorem 3.7). Also, we will provide a sufficient condition for $P^\bullet$ to be a direct summand of a tilting complex provided that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ (Theorem 3.10).

2 Noether algebras

This section is a joint work with M. Hoshino. Throughout the rest of this paper R is a commutative Noether ring and A is a Noether R-algebra and throughout the rest of this section we assume that $R_{\mathfrak{p}}$ is a Gorenstein ring for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and that $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A, R)$.

Remark 2.1. Under the assumption that $R_{\mathfrak{p}}$ is a Gorenstein ring for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$, the following are equivalent:

(1) $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$.

(2) A is a maximal Cohen-Macaulay R-module.

(3) A has Gorenstein dimension 0 as an R-module.

We refer to [Mi] for the definition of tilting modules.

Theorem 2.2. *The following are equivalent.*

(1) proj dim $\Omega_A \leq 1$.

(2) proj dim ${}_A\Omega \leq 1$.

(3) Ω *is a tilting module with* proj dim $\Omega_A \leq 1$ *and* proj dim ${}_A\Omega \leq 1$.

Proposition 2.3. *Assume that R is Gorenstein local. Then we have*

$$\text{inj dim } {}_AA = \text{proj dim } \Omega_A + \dim R.$$

Corollary 2.4. *Assume that R is a Gorenstein local ring. Then the following are equivalent:*

(1) inj dim $A_A \leq \dim R + 1$.

(2) inj dim ${}_AA \leq \dim R + 1$.

We assume that $\sup\{\mathrm{ht}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$. Take a minimal injective resolution $R \to I^\bullet$ in Mod-R and set $V^\bullet = \mathrm{Hom}^\bullet_R(A, I^\bullet)$. Note that $\mathrm{H}^0(V^\bullet) \cong \Omega$ and $\mathrm{H}^i(V^\bullet) = 0$ for $i \neq 0$. We refer to [Ri] for the definition of tilting complexes.

Proposition 2.5 ([Ab]). *The following are equivalent.*

(1) inj dim ${}_A A =$ inj dim A_A.

(2) There exists a quasi-isomorphism $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ a tilting complex such that $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$.

(3) There exists a quasi-isomorphism $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\mathrm{op}})$ with $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\mathrm{op}}})$ a tilting complex such that $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A^{\mathrm{op}})}(Q^\bullet)^{\mathrm{op}}$.

(4) There exists a quasi-isomorphism $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\mathrm{op}})$ with $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\mathrm{op}}})$.

3 Partial tilting complexes

This section is based on [Ko].

Definition 3.1. a complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ is said to be a partial tilting complex if there exists a complex $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $P^\bullet \oplus Q^\bullet$ is a tilting complex.

Definition 3.2. For a complex $X^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we set

$$\mathrm{a}(X^\bullet) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i(X^\bullet) \neq 0\}.$$

Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$ and set $l = \mathrm{a}(P^\bullet)$. We may assume that $P^i = 0$ unless $0 \leq i \leq l$. We construct a sequence of cochain complexes $Q_0^\bullet, Q_1^\bullet, \cdots$ in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as follows. Set $Q_0^\bullet = A$. Let $k \geq 0$ and assume that $Q_0^\bullet, \cdots, Q_k^\bullet$ have been constructed. Then by the follwoing Lemma we have a distinguished triangle in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$

$$\Delta_k : Q_{k+1}^\bullet \xrightarrow{g_{k+1}} \overset{n_k}{\oplus} P^\bullet \xrightarrow{f_k} Q_k^\bullet \to$$

with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, f_k)$ epic. Throughout this section, we will keep the notation above.

Lemma 3.3. *For any $X^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Y^\bullet \in \mathcal{K}(\mathcal{P}_A)$ we have a cochain map*

$$f : \overset{n}{\oplus} X^\bullet \to Y^\bullet$$

with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(X^\bullet, f)$ epic.

Definition 3.4 ([Ri]). Let $\mathcal{K}$ be a triangulated category. Then a full subcategory $\mathcal{H}$ of $\mathcal{K}$ is said to generate $\mathcal{K}$ as a triangulated category if there is no proper épaisse subcategory of $\mathcal{K}$ containing $\mathcal{H}$. For a full subcategory $\mathcal{H}$ of $\mathcal{K}$ we denote by $\langle \mathcal{H} \rangle$ the épaisse subcategory of $\mathcal{K}$ generated by $\mathcal{H}$.

Definition 3.5 ([KV]). A complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is said to be a silting complex if the following conditions are satisfied:

(1) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$.

(2) $\mathrm{add}(P^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category.

Lemma 3.6. *Assume that $l \geq 2$. For any $k \geq l$ the following are equivalent.*

(1) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q_k^\bullet, Q_k^\bullet[i]) = 0$ *for* $1 \leq i < l$.

(2) $\mathrm{H}^l(f_i)$ *is epic for* $1 \leq i < l$.

(3) $\mathrm{a}(Q_l^\bullet) \leq l$.

(4) $\mathrm{a}(Q_k^\bullet) \leq k$.

Theorem 3.7. *Assume that* $\mathrm{a}(Q_l^\bullet) \leq l$. *Then* $P^\bullet \oplus Q_k^\bullet$ *is a silting complex for all* $k \geq l$.

Theorem 3.8. *Assume that* $\mathrm{H}^i(P^\bullet) = 0$ *for* $i \neq l$ *and* $\mathrm{a}(Q_l^\bullet) \leq l$. *Then* $P^\bullet \oplus Q_l^\bullet$ *is a tilting complex with* $\mathrm{H}^i(P^\bullet \oplus Q_l^\bullet) = 0$ *for* $i \neq l$.

We will provide a sufficient condition for $P^\bullet \oplus Q_l^\bullet$ to be a tilting complex. Set $D = \mathrm{Hom}_R(-, R)$ and $\nu = D \circ \mathrm{Hom}_A(-, A)$.

Lemma 3.9. *Let $k \geq 0$ and assume that* $\mathrm{Ext}_R^i(A, R) = 0$ *for* $1 \leq i < k - 1 + \max\{l - 1, 0\}$. *Then we have* $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q_k^\bullet, \nu P^\bullet[i]) = 0$ *for* $i < 0$.

Theorem 3.10. *Assume that* $P^\bullet \in \mathrm{add}(\nu P^\bullet)$ *and* $\mathrm{a}(Q_l^\bullet) \leq l$. *Let $k \geq l$ and assume that* $\mathrm{Ext}_R^i(A, R) = 0$ *for* $1 \leq i < k - 1 + \max\{l - 1, 0\}$. *Then* $P^\bullet \oplus Q_k^\bullet$ *is a tilting complex.*

Assume that $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a partial tilting complex and set $B = \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$.

Proposition 3.11. *Assume that R is a complete local ring. Then $P^\bullet$ is a tilting complex whenever* $\mathrm{rank}\ \mathrm{K}_0(A) = \mathrm{rank}\ \mathrm{K}_0(B)$.

References

[Ab] H. Abe, Noetherian algebras of finite selfinjective dimension, Comm. Algebra **36** (2008), no. 2, 493–507.

[Gr] A. Grothendieck, Groupes des classes des catègories abeliennes et triangulées. complexes parfaits, in: *Seminaire de Geometrie Algebrique du Bois-Marie* (SGA 5), 351–371, Lecture Notes in Math., **589**, Springer, Berlin, 1977.

[KV] B. Keller and D. Vossieck, Aisles in derived categories, Bull. Soc. Math. Belg. **40** (1988), no. 2, 239–253.

[Ko] H. Koga, On partial tilting complexes, Comm. Algebra, to appear.

[Mi] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z. **193** (1986), no. 1, 113–146.

[Ri] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.

[Za] A. Zaks, Injective dimension of semi-primary rings, J. Algebra 13 (1969), 73-86.