On the symmetry of selfinjective dimension

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Abstract

Let $R$ be a commutative Noether ring and $A$ a Noether $R$-algebra. Set $\Omega = \text{Hom}_R(A, R)$. We will show that $\text{proj dim}_A \Omega \leq 1$ if and only if $\text{proj dim} \Omega_A \leq 1$ under some conditions. Also, we will show that $\text{inj dim}_A A \leq \dim R + 1$ if and only if $\text{inj dim} A_A \leq \dim R + 1$ under some conditions. Assume that $\text{proj dim} \Omega_A < \infty$. Let $P^* \to \Omega$ be a projective resolution with $P^* \in \mathbb{K}^b(P_A)$. We will provide a sufficient condition for $P^*$ to be a partial tilting complex.

1 Introduction and notation

1.1 Notation

Let $A$ be a ring. We denote by $\text{Mod-}A$ the category of right $A$-modules. We denote by $A^{\text{op}}$ the opposite ring of $A$ and consider left $A$-modules as right $A^{\text{op}}$-modules. Sometimes, we use the notation $M_A$ (resp., $_AM$) to stress that the module $M$ considered is a right (resp., left) $A$-module. We denote by $\mathcal{P}_A$ the full subcategory of $\text{Mod-}A$ consisting of finitely generated projective modules. For an additive category $\mathcal{A}$ we denote by $\mathcal{K}(\mathcal{A})$ the homotopy category of cochain complexes over $\mathcal{A}$ and by $\mathcal{K}^b(\mathcal{A})$ the full triangulated subcategories of $\mathcal{K}(\mathcal{A})$ consisting of bounded complexes. We refer to [Gr] for the definition of Grothendieck groups and denote by $K_0(A)$ the Grothendieck group of $A$. For an object $X$ in an additive category $\mathcal{A}$ we denote by $\text{add}(X)$ the full subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of copies of $X$. We denote by $\text{Hom}^*(-, -)$ the associated single complex of the double hom complex. For a cochain complex $X^*$ over an abelian category $\mathcal{A}$ we denote by $H^i(X^*)$ the $i$th cohomology of $X^*$. We consider modules as complexes concentrated in degree zero.
Let $R$ be a commutative ring. We denote by $\dim R$ the Krull dimension of $R$. We denote by $\text{Spec}(R)$ the set of prime ideals of $R$. For any $p \in \text{Spec}(R)$ we denote by $(-)_p$ the localization at $p$. For any $M \in \text{Mod}_R$ we set $\text{Supp}_R(M) = \{p \in \text{Spec}(R) \mid M_p \neq 0\}$.

1.2 Introduction

Let $A$ be a Noether ring, i.e., $A$ is a left and right Noether ring. Does it hold true that $\text{inj dim } A < \infty$ implies $\text{inj dim } A_A < \infty$? In [Za, Lemma A] Zaks showed that if $\text{inj dim } A_A < \infty$ and $\text{inj dim } A_A < \infty$ then $\text{inj dim } A^A = \text{inj dim } A_A$. This problem is still open.

We will consider the case where $R$ is a commutative Noether ring and $A$ is a Noether $R$-algebra, i.e., $A$ is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of $A$ and $A$ is finitely generated as an $R$-module. Note that $A$ is left and right noetherian. Assume that $R$ and $A$ satisfy the following conditions: (1) $R_p$ is a Gorenstein ring for all $p \in \text{Supp}_R(A)$; (2) $\text{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \text{Hom}_R(A, R)$. Then we will show that $\text{proj dim } A^A \leq 1$ if and only if $\text{proj dim } \Omega_A \leq 1$ (Theorem 2.2). Assume further that $R$ is a Gorenstein local ring. Then we will provide a formula of selfinjective dimension (Proposition 2.3) and show the symmetry of selfinjective dimension (Corollary 2.4). Assume further that $\sup \{\text{ht } p \mid p \in \text{Supp}_R(A)\} < \infty$. We know from [Ab, Theorem 3.9] that the following are equivalent: (1) $\text{inj dim } A^A = \text{inj dim } A_A < \infty$; (2) $\text{proj dim } A_A = \text{proj dim } A^A \leq \infty$; (3) $\Omega$ is a tilting module. Take a projective resolution $P^* \to \Omega$ in $\text{mod}_A$. If $P^*$ is a partial tilting complex, i.e., $P^*$ is a direct summand of a tilting complex, then we can conclude that $\text{inj dim } A^A < \infty$ if and only if $\text{inj dim } A_A < \infty$ (cf. Proposition 3.11). So we ask when $P^*$ is a partial tilting complex.

Let $P^* \in \mathcal{K}^b(P_A)$ with $\text{Hom}_{\mathcal{K}(\text{Mod}_A)}(P^*, P^*[i]) = 0$ for $i > 0$. We will provide a sufficient condition for $P^*$ to be a direct summand of a silting complex (Theorem 3.7). Also, we will provide a sufficient condition for $P^*$ to be a direct summand of a tilting complex provided that $\text{Hom}_{\mathcal{K}(\text{Mod}_A)}(P^*, P^*[i]) = 0$ for $i \neq 0$ (Theorem 3.10).
2 Noether algebras

This section is a joint work with M. Hoshino. Throughout the rest of this paper $R$ is a commutative Noether ring and $A$ is a Noether $R$-algebra and throughout the rest of this section we assume that $R_p$ is a Gorenstein ring for all $p \in \text{Supp}_R(A)$ and that $\text{Ext}_R^i(A, R) = 0$ for $i \neq 0$. Set $\Omega = \text{Hom}_R(A, R)$.

**Remark 2.1.** Under the assumption that $R_p$ is a Gorenstein ring for all $p \in \text{Supp}_R(A)$, the following are equivalent:

1. $\text{Ext}_R^i(A, R) = 0$ for $i \neq 0$.
2. $A$ is a maximal Cohen-Macaulay $R$-module.
3. $A$ has Gorenstein dimension 0 as an $R$-module.

We refer to [Mi] for the definition of tilting modules.

**Theorem 2.2.** The following are equivalent.

1. $\text{proj dim } \Omega_A \leq 1$.
2. $\text{proj dim } A\Omega \leq 1$.
3. $\Omega$ is a tilting module with $\text{proj dim } \Omega_A \leq 1$ and $\text{proj dim } A\Omega \leq 1$.

**Proposition 2.3.** Assume that $R$ is Gorenstein local. Then we have

$$\text{inj dim } A_A = \text{proj dim } \Omega_A + \dim R.$$ 

**Corollary 2.4.** Assume that $R$ is a Gorenstein local ring. Then the following are equivalent:

1. $\text{inj dim } A_A \leq \dim R + 1$.
2. $\text{inj dim } A_A \leq \dim R + 1$.

We assume that $\sup \{ \text{ht } p | p \in \text{Supp}_R(A) \} < \infty$. Take a minimal injective resolution $R \to I^\bullet$ in $\text{Mod}-R$ and set $V^\bullet = \text{Hom}_R^*(A, I^\bullet)$. Note that $H^0(V^\bullet) \cong \Omega$ and $H^i(V^\bullet) = 0$ for $i \neq 0$. We refer to [Ri] for the definition of tilting complexes.

**Proposition 2.5 ([Ab]).** The following are equivalent.
(1) $\text{inj dim } A = \text{inj dim } A_A$.

(2) There exists a quasi-isomorphism $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^b(\mathcal{P}_A)$ a tilting complex such that $A \cong \text{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$.

(3) There exists a quasi-isomorphism $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\text{op}})$ with $Q^\bullet \in \mathcal{K}^b(\mathcal{P}_{A^{\text{op}}})$ a tilting complex such that $A \cong \text{End}_{\mathcal{K}(\text{Mod-}A^{\text{op}})}(Q^\bullet)^{\text{op}}$.

(4) There exists a quasi-isomorphism $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^b(\mathcal{P}_A)$ and $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\text{op}})$ with $Q^\bullet \in \mathcal{K}^b(\mathcal{P}_{A^{\text{op}}})$.

3 Partial tilting complexes

This section is based on [Ko].

Definition 3.1. A complex $P^\bullet \in \mathcal{K}^b(\mathcal{P}_A)$ with $\text{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ is said to be a partial tilting complex if there exists a complex $Q^\bullet \in \mathcal{K}^b(\mathcal{P}_A)$ such that $P^\bullet \oplus Q^\bullet$ is a tilting complex.

Definition 3.2. For a complex $X^\bullet \in \mathcal{K}^b(\mathcal{P}_A)$ we set

$$a(X^\bullet) = \sup \{ i \in \mathbb{Z} \mid H^i(X^\bullet) \neq 0 \}.$$

Let $P^\bullet \in \mathcal{K}^b(\mathcal{P}_A)$ with $\text{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$ and set $l = a(P^\bullet)$. We may assume that $P^i = 0$ unless $0 \leq i \leq l$. We construct a sequence of cochain complexes $Q^\bullet_0, Q^\bullet_1, \ldots$ in $\mathcal{K}^b(\mathcal{P}_A)$ as follows. Set $Q^\bullet_0 = A$. Let $k \geq 0$ and assume that $Q^\bullet_0, \ldots, Q^\bullet_k$ have been constructed. Then by the following Lemma we have a distinguished triangle in $\mathcal{K}^b(\mathcal{P}_A)$

$$\Delta_k : Q^\bullet_{k+1} \xrightarrow{g_{k+1}} Q^\bullet_k \oplus P^\bullet \xrightarrow{f_k} Q^\bullet_k \to$$

with $\text{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, f_k)$ epic. Throughout this section, we will keep the notation above.

Lemma 3.3. For any $X^\bullet \in \mathcal{K}^b(\mathcal{P}_A)$ and $Y^\bullet \in \mathcal{K}(\mathcal{P}_A)$ we have a cochain map

$$f : \oplus X^\bullet \to Y^\bullet$$

with $\text{Hom}_{\mathcal{K}(\text{Mod-}A)}(X^\bullet, f)$ epic.
Definition 3.4 ([Ri]). Let $\mathcal{K}$ be a triangulated category. Then a full subcategory $\mathcal{H}$ of $\mathcal{K}$ is said to generate $\mathcal{K}$ as a triangulated category if there is no proper épaisse subcategory of $\mathcal{K}$ containing $\mathcal{H}$. For a full subcategory $\mathcal{H}$ of $\mathcal{K}$ we denote by $\langle \mathcal{H} \rangle$ the épaisse subcategory of $\mathcal{K}$ generated by $\mathcal{H}$.

Definition 3.5 ([KV]). A complex $P^\bullet \in \mathcal{K}^b(P_A)$ is said to be a silting complex if the following conditions are satisfied:

1. $\text{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$.
2. $\text{add}(P^\bullet)$ generates $\mathcal{K}^b(P_A)$ as a triangulated category.

Lemma 3.6. Assume that $l \geq 2$. For any $k \geq l$ the following are equivalent.

1. $\text{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet_k, Q^\bullet_k[i]) = 0$ for $1 \leq i < l$.
2. $H^i(f_i)$ is epic for $1 \leq i < l$.
3. $a(Q^\bullet_k) \leq l$.
4. $a(Q^\bullet_k) \leq l$.

Theorem 3.7. Assume that $a(Q^\bullet_l) \leq l$. Then $P^\bullet \oplus Q^\bullet_k$ is a silting complex for all $k \geq l$.

Theorem 3.8. Assume that $H^i(P^\bullet) = 0$ for $i \neq l$ and $a(Q^\bullet_l) \leq l$. Then $P^\bullet \oplus Q^\bullet_l$ is a tilting complex with $H^i(P^\bullet \oplus Q^\bullet_l) = 0$ for $i \neq l$.

We will provide a sufficient condition for $P^\bullet \oplus Q^\bullet_l$ to be a tilting complex.

Set $D = \text{Hom}_R(\mathcal{K}^b(P_A), \mathcal{K}^b(P_A))$ and $\nu = D \circ \text{Hom}_A(\mathcal{K}^b(P_A), \mathcal{K}^b(P_A))$.

Lemma 3.9. Let $k \geq 0$ and assume that $\text{Ext}^i_R(A, R) = 0$ for $1 \leq i < k - 1 + \max\{l - 1, 0\}$. Then we have $\text{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet_k, \nu P^\bullet[i]) = 0$ for $i < 0$.

Theorem 3.10. Assume that $P^\bullet \in \text{add}(\nu P^\bullet)$ and $a(Q^\bullet_l) \leq l$. Let $k \geq l$ and assume that $\text{Ext}^i_R(A, R) = 0$ for $1 \leq i < k - 1 + \max\{l - 1, 0\}$. Then $P^\bullet \oplus Q^\bullet_k$ is a tilting complex.

Assume that $P^\bullet \in \mathcal{K}^b(P_A)$ is a partial tilting complex and set $B = \text{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$.

Proposition 3.11. Assume that $R$ is a complete local ring. Then $P^\bullet$ is a tilting complex whenever $\text{rank } K_0(A) = \text{rank } K_0(B)$.

5
References


