SMOOTH FANO POLYTOPES ARISING FROM FINITE POSETS

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[This study is a joint work with Prof. Takayuki Hibi.]

Abstract. Gorenstein Fano polytopes arising from finite posets will be introduced. Then we study the problem of which posets yield smooth Fano polytopes.

Introduction

An integral (or lattice) polytope is a convex polytope all of whose vertices have integer coordinates. Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d.

- We say that $\mathcal{P}$ is a *Fano polytope* if the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior of $\mathcal{P}$.
- A Fano polytope $\mathcal{P}$ is called *terminal* if each integer point belonging to the boundary of $\mathcal{P}$ is a vertex of $\mathcal{P}$.
- A Fano polytope $\mathcal{P}$ is called *canonical* if $\mathcal{P}$ is not terminal, i.e., there is an integer point belonging to the boundary of $\mathcal{P}$ which is not a vertex of $\mathcal{P}$.
- A Fano polytope is called *Gorenstein* if its dual polytope is integral. (Recall that the dual polytope $\mathcal{P}^\vee$ of a Fano polytope $\mathcal{P}$ is the convex polytope which consists of those $x \in \mathbb{R}^d$ such that $\langle x, y \rangle \leq 1$ for all $y \in \mathcal{P}$, where $\langle x, y \rangle$ is the usual inner product of $\mathbb{R}^d$.)
- A $\mathbb{Q}$-*factorial Fano polytope* is a simplicial Fano polytope, i.e., a Fano polytope each of whose faces is a simplex.
- A *smooth Fano polytope* is a Fano polytope such that the vertices of each facet form a $\mathbb{Z}$-basis of $\mathbb{Z}^d$.

Thus in particular a smooth Fano polytope is $\mathbb{Q}$-factorial Fano, Gorenstein and terminal.

Øbro [0] succeeded in finding an algorithm which yields the classification list of the smooth Fano polytopes for given d. It is proved in Casagrande [0] that the number of vertices of a Gorenstein $\mathbb{Q}$-factorial Fano polytope is at most $3d$ if d is even, and at most $3d-1$ if d is odd. B. Nill and M. Øbro [0] classified the Gorenstein $\mathbb{Q}$-factorial Fano polytopes of dimension d with $3d-1$ vertices. Gorenstein Fano polytopes are classified when $d \leq 4$ by Kreuzer and Skarke [0], [0]. The study on the classification of terminal or canonical Fano polytopes was done by Kasprzyk [0].

In this talk, given a finite poset P we introduce a terminal Gorenstein Fano polytope $\mathcal{X}_P$. Then we study the problem of which posets yield $\mathbb{Q}$-factorial Fano polytopes. Finally, it turns out that the Fano polytope $\mathcal{X}_P$ is smooth if and only if $\mathcal{X}_P$ is $\mathbb{Q}$-factorial.

1. Fano polytopes arising from finite posets

Let $P = \{y_1, \ldots, y_d\}$ be a finite poset and $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$, where $\hat{0}$ (resp. $\hat{1}$) is a unique minimal (resp. maximal) element of $\hat{P}$ with $\hat{0} \notin P$ (resp. $\hat{1} \notin P$). Let $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. We say that $e = \{y_i, y_j\}$, where $0 \leq i,\, j \leq d+1$ with $i \neq j$, is an *edge* of $\hat{P}$ if e is an edge of the Hasse diagram of $\hat{P}$. (The Hasse diagram of a finite poset can be regarded as a finite undirected graph.) In other words, $e = \{y_i, y_j\}$ is an edge of $\hat{P}$ if y_i and y_j are comparable in $\hat{P}$, say, $y_i < y_j$, and there is no $z \in P$ with $y_i < z < y_j$.

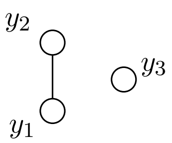
Definition 1.1. Let $\hat{P} = \{y_0, y_1, \ldots, y_d, y_{d+1}\}$ be a finite poset with $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. Let $\mathbf{e}_i, i = 1, \ldots, d$, denote the ith canonical unit coordinate vector of $\mathbb{R}^d$. Given an edge $e = \{y_i, y_j\}$ of $\hat{P}$ with $y_i < y_j$, we define $\rho(e) \in \mathbb{R}^d$ by setting

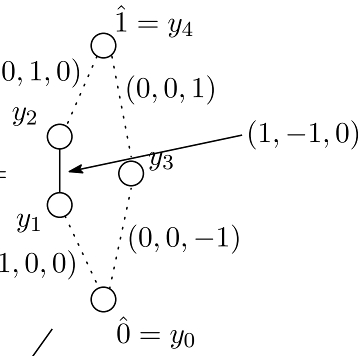
$$\rho(e) = \mathbf{e}_i - \mathbf{e}_j \quad \text{for} \quad 0 \leq i, j \leq d+1,$$

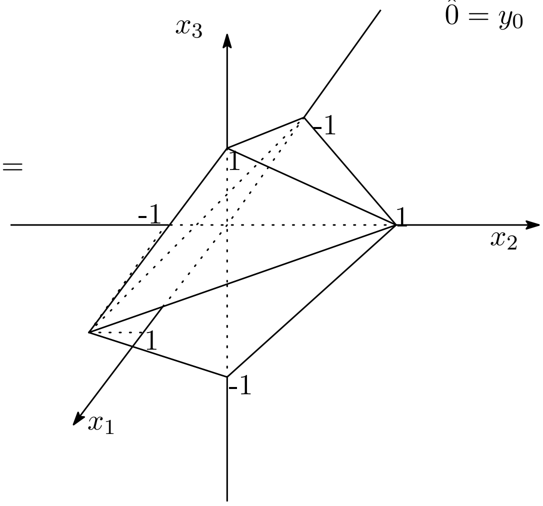
where $\mathbf{e}_0 = \mathbf{e}_{d+1} = 0$. Moreover, we write $\mathcal{X}_P \subset \mathbb{R}^d$ for the convex hull of the finite set

$$\{\, \rho(e) : \, e \text{ is an edge of } \hat{P}\,\}.$$

Example 1.2. Let $P = \{y_1, y_2, y_3\}$ be the finite poset with the partial order $y_1 < y_2$. Then $\hat{P}$ together with $\rho(e)$'s and $\mathcal{X}_P$ are drawn below:

$P =$ 

$\hat{P} =$ 

$\mathcal{X}_P =$ 

Lemma 1.3. *The convex polytope $\mathcal{X}_P$ is a terminal Gorenstein Fano polytope.*

2. When is $\mathcal{X}_P$ $\mathbb{Q}$-factorial?

Let $P = \{y_1, \ldots, y_d\}$ be a finite poset and $\hat{P} = P \cup \{y_0, y_{d+1}\}$, where $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. A sequence $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ is called a *path* in $\hat{P}$ if Γ is a path in the Hasse diagram of $\hat{P}$. In other words, $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ is a path in $\hat{P}$ if $y_{i_j} \neq y_{i_k}$ for all $1 \leq j < k \leq m$ and if $\{y_{i_j}, y_{i_{j+1}}\}$ is an edge of $\hat{P}$ for all $1 \leq j \leq m-1$. In particular, if $\{y_{i_1}, y_{i_m}\}$ is also an edge of $\hat{P}$, Γ is called a *cycle*. The *length* of a path $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ is $\ell(\Gamma) = m-1$, while the length of a cycle is m.

A path $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_{m+1}})$ is called *ranked* if

$$\sharp\{\, j : y_{i_j} < y_{i_{j+1}}, 1 \leq j \leq m \,\} = \sharp\{\, k : y_{i_k} > y_{i_{k+1}}, 1 \leq k \leq m \,\}.$$

Given a ranked path $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$, there exists a unique function

$$\mu_\Gamma : \{y_{i_1}, y_{i_2}, \ldots, y_{i_m}\} \to \{0, 1, 2, \ldots\}$$

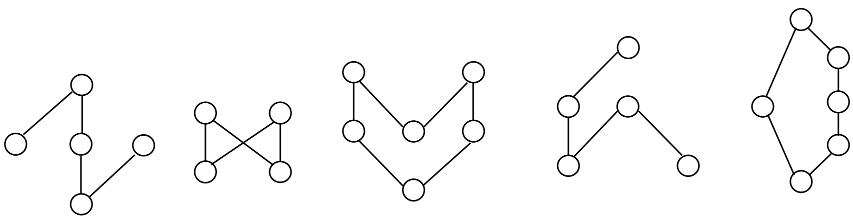
such that

- $\mu_\Gamma(y_{i_{j+1}}) = \mu_\Gamma(y_{i_j}) + 1$ (resp. $\mu_\Gamma(y_{i_j}) = \mu_\Gamma(y_{i_{j+1}}) + 1$) if $y_{i_j} < y_{i_{j+1}}$ (resp. $y_{i_j} > y_{i_{j+1}}$);
- $\min\{\mu_\Gamma(y_{i_1}), \mu_\Gamma(y_{i_2}), \ldots, \mu_\Gamma(y_{i_m})\} = 0$.

In particular, Γ is ranked if and only if $\mu_\Gamma(y_{i_1}) = \mu_\Gamma(y_{i_m})$.

Similary, a ranked cycle is defined. Given a ranked cycle C, there exists a unique function μ_C which is defined the same way as above.

Example 2.1. Among the two paths and three cycles drawn below, each of one path and two cycles on the left-hand side is ranked; none of one path and one cycle on the right-hand side is ranked.



Let P be a finite poset. A subset Q of P is called a *chain* of P if Q is a totally ordered subset of P. The *length* of a chain Q is $\ell(Q) = \sharp(Q) - 1$. A chain Q of P is *saturated* if $x, y \in Q$ with $x < y$, then there is no $z \in P$ with $x < z < y$. A *maximal* chain of $\hat{P}$ is a saturated chain Q of $\hat{P}$ with $\{\hat{0}, \hat{1}\} \subset Q$. Let $y, z \in P$ with $y < z$. The *distance* of y and z in $\hat{P}$ is the smallest integer s for which there is a saturated chain $Q = \{z_0, z_1, \ldots, z_s\}$ with

$$y = z_0 < z_1 < \cdots < z_s = z.$$

Let $\mathrm{dist}_{\hat{P}}(y, z)$ denote the distance of y and z in $\hat{P}$.

We now come to the main theorem.

Theorem 2.2. *Let $P = \{y_1, \ldots, y_d\}$ be a finite poset and $\hat{P} = P \cup \{y_0, y_{d+1}\}$, where $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. Then the following conditions are equivalent:*

(i) $\mathcal{X}_P$ is $\mathbb{Q}$-factorial;

(ii) $\mathcal{X}_P$ *is smooth;*

(iii) $\hat{P}$ *possesses no ranked path* $\Gamma = (y_{i_1}, \ldots, y_{i_m})$ *with* $y_{i_1} = y_0$ *and* $y_{i_m} = y_{d+1}$ *such that*

$$\mu_\Gamma(y_{i_a}) - \mu_\Gamma(y_{i_b}) \leq \mathrm{dist}_{\hat{P}}(y_{i_b}, y_{i_a}) \tag{1}$$

for all $1 \leq a, b \leq m$ *with* $y_{i_b} < y_{i_a}$*, and no ranked cycle* $C = (y_{i_1}, \ldots, y_{i_m})$ *with* $\{y_0, y_{d+1}\} \not\subset \{y_{i_1}, y_{i_2}, \ldots, y_{i_m}\}$ *such that*

$$\mu_C(y_{i_a}) - \mu_C(y_{i_b}) \leq \mathrm{dist}_{\hat{P}}(y_{i_b}, y_{i_a}) \tag{2}$$

for all $1 \leq a, b \leq m$ *with* $y_{i_b} < y_{i_a}$*, and*

$$\mu_C(y_{i_a}) - \mu_C(y_{i_b}) \leq \mathrm{dist}_{\hat{P}}(y_0, y_{i_a}) + \mathrm{dist}_{\hat{P}}(y_{i_b}, y_{d+1}) \tag{3}$$

for all $1 \leq a, b \leq m$.

Recall that a finite poset P is *pure* if all maximal chains of $\hat{P}$ have the same length.

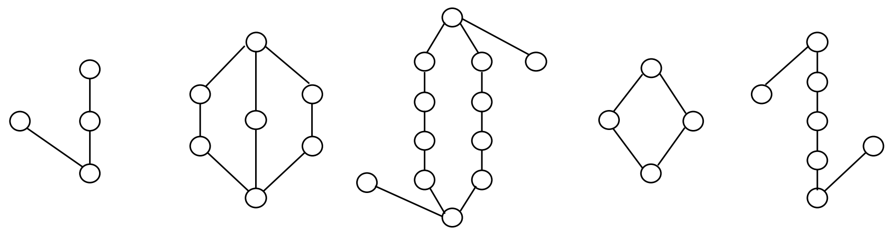
Corollary 2.3. *Suppose that a finite poset* P *is pure. Then the following conditions are equivalent:*

(i) $\mathcal{X}_P$ *is* $\mathbb{Q}$*-factorial;*

(ii) $\mathcal{X}_P$ *is smooth;*

(iii) P *is a disjoint union of chains.*

Example 2.4. Among the five posets drawn below, each of the three posets on the left-hand side yields a $\mathbb{Q}$-factorial Fano polytope; none of the two posets on the right-hand side yields a $\mathbb{Q}$-factorial Fano polytope.



Let P and P' be finite partially ordered sets. Then one can verify easily that $\mathcal{X}_P$ is isomorphic with $\mathcal{X}_{P'}$ as a convex polytope if and only if P is isomorphic with P' or with the dual finite partially ordered set of P' as a finite partially ordered set.

On the following table drawn below, the number of finite partially ordered sets with $d(\leq 8)$ elements, up to isomorphic and up to isomorphic with dual finite partially ordered sets, is written in the second row. Moreover, among those, the number of finite partially ordered sets constructing smooth Fano polytopes is written in the third row.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
Posets	1	2	4	12	39	184	1082	8746
Smooth	1	2	3	6	12	31	83	266

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