FROBENIUS CONDITION ON A PRETRIANGULATED CATEGORY

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As shown in [H], from any Frobenius exact category, we can construct a triangulated category as a stable category. On the other hand, it was shown in [IY] that if a pair of subcategories $\mathcal{D} \subseteq \mathcal{Z}$ in a triangulated category satisfies certain conditions (namely, $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair), then $\mathcal{Z}/\mathcal{D}$ becomes a triangulated category. In this talk, we will make a simultaneous generalization of these two constructions.

We define a *pretriangulated category* as a quintet $(\mathcal{C}, \Sigma, \Omega, \lhd, \rhd)$ of an additive category $\mathcal{C}$, additive endofunctors $\Omega, \Sigma \colon \mathcal{C} \to \mathcal{C}$, and classes of right and left triangles $\rhd$, $\lhd$, satisfying some conditions similar to those in [BR]. We often represent a pretriangulated category simply by $\mathcal{C}$. With this definition, a triangulated category is a pretriangulated category with $\Omega \cong \Sigma^{-1}$, and any abelian category can be regarded as a pretriangulated category satisfying $\Sigma = \Omega = 0$.

In a pretriangulated category, we can define a *short exact sequence*

$$\Omega C \to A \to B \to C \to \Sigma A,$$

which generalize a short exact sequence in an abelian category and a distinguished triangle in a triangulated category. With this definition, we can consider *extension-closedness* of a subcategory $\mathcal{Z} \subseteq \mathcal{C}$.

For a triplet $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ of a pretriangulated category $\mathcal{C}$, an extension-closed subcategory $\mathcal{Z} \subseteq \mathcal{C}$ and a subcategory $\mathcal{D} \subseteq \mathcal{Z}$ satisfying $\mathcal{C}(\mathcal{Z}, \Sigma\mathcal{D}) = \mathcal{C}(\Omega\mathcal{D}, \mathcal{Z}) = 0$, we can define the class of *injective objects* $\mathcal{I}$ and that of *projective objects* $\mathcal{P}$.

	Happel's construction [H]	Iyama and Yoshino's construction [IY]
$\mathcal{C}$	abelian category	triangulated category
$\mathcal{Z}$	exact subcategory	extension-closed subcategory
$\mathcal{D}$	$\mathcal{Z} = \mathcal{D}$	$(\mathcal{Z}, \mathcal{Z})$: $\mathcal{D}$-mutation pair
$\mathcal{I}$	injective objects	$\mathcal{I}$=$\mathcal{D}$
$\mathcal{P}$	projective objects	$\mathcal{P}$=$\mathcal{D}$

We impose 'Frobenius condition' (including $\mathcal{I} = \mathcal{P} =: \mathcal{F}$) on the triplet $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$, and will show the following.

Theorem . Let $\mathcal{C}$ be a pretriangulated category, $\mathcal{Z} \subseteq \mathcal{C}$ be an extension-closed subcategory, and $\mathcal{D} \subseteq \mathcal{Z}$ be a triangulator. If $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius, then $\mathcal{Z}/\mathcal{F}$ becomes a triangulated category.

References

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