On the symmetry of selfinjective dimension

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Abstract

Let A be a Noether ring, i.e., A is a left and right Noether ring. Does it hold true that inj dim ${}_AA < \infty$ implies inj dim $A_A < \infty$? In [Za, Lemma A] Zaks showed that if inj dim ${}_AA < \infty$ and inj dim $A_A < \infty$ then inj dim ${}_AA =$ inj dim A_A. This problem is still open. We consider the case where R is a commutative Noether ring and A is a Noether R-algebra. Assume R and A satisfy the following conditions: (1) $R_{\mathfrak{p}}$ is a Gorenstein ring for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$; (2) $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A, R)$. Then we show proj dim ${}_A\Omega \leq 1$ if and only if proj dim $\Omega_A \leq 1$. Assume further that R is a Gorenstein local ring. Then we provide a formula of selfinjective dimension and show the symmetry of selfinjective dimension. Assume further that $\sup \{\mathrm{ht}\, \mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$. We see from [Ab, Theorem 3.6] that the following are equivalent: (1) inj dim ${}_AA =$ inj dim $A_A < \infty$; (2) proj dim $\Omega_A =$ proj dim ${}_A\Omega < \infty$; (3) Ω is a tilting module. Take a projective resolution $P^\bullet \to \Omega$ in mod-A. If $P^\bullet$ is a partial tilting complex, i.e., $P^\bullet$ is a direct summand of a tilting complex, then we can conlude that inj dim ${}_AA < \infty$ if and only if inj dim $A_A < \infty$. So we ask when $P^\bullet$ is a partial tilting complex. More generally, let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$. We provide a sufficient condition for $P^\bullet$ to be a direct summand of a silting complex. Also, we provide a sufficient condition for $P^\bullet$ to be a direct summand of a tilting complex provided that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$.

References

[Ab] H. Abe, Noetherian algebras of finite selfinjective dimension, Comm. Algebra 36, (2008), 493–507.

[Za] A. Zaks, Injective dimension of semi-primary rings, J. Algebra 13 (1969), 73-86.